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ONE NON-LINEAR PROBLEM WITH A FREE BOUNDARY

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ABSTRACT

Let G_z denote the region between the unit circle Γ and the free boundary γ. Find the function ψ so that ψ is harmonic within G_z and continuous on $G_z + \Gamma + \gamma$; $\psi = 0$ on Γ; ψ = const on γ; grad $\psi = Q$ on γ, with $Q(x, y)$ given.

1. <u>Formulation of the Problem.</u> Let $|z| \leq 1$, $z = x + iy$ hold with- /979*
in a unit circle; let the (sufficiently continuous) function $Q(x, y) > 0$ be given. It is necessary to determine the continuous curve γ located within the circle $|z| < 1$, so that in doubly-connected region G_z, which is limited by γ and a unit circle Γ: $|z| = 1$, the following conditions are fulfilled: 1) the function $\psi(x, y)$ exists which is harmonic within G_z and continuous in $G_z + \gamma + \Gamma$; 2) $\psi = 0$ on Γ; 3) $\psi = c_1$, $c_1 = \text{const} \neq 0$ on γ; 4) $|\text{grad}\ \psi| = Q$ on γ .

Since the formulation of the problem is invariant with respect to the conformal mapping, the case of an arbitrary curve Γ which defines the doubly-connected region may be readily reduced to the case under consideration.

This problem arises in hydrodynamics where special assumptions regarding Γ and Q are formulated (the theory of waves in a heavy liquid, jet flows), and in these cases, considerable progress has been made regarding this problem.

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* Note: Numbers in the margin indicate pagination in the original foreign text.

Thus, A. I. Nekrasov was the first to show the existence of periodic waves in a heavy liquid (1921) [see, for example, (Ref. 4), page 358]. M. A. Lavrent'yev (Ref. 3) proved the existence of a single wave (1946). Recent studies on this problem have been compiled in a book (Ref. 5). In the formulation we are considering, the problem was studied by Beurling (Ref. 2), which presented a classification of the possible cases, indicated certain criteria for the solution, and also presented sufficient conditions of uniqueness. This article applies the analytical method to study the problem, presents criteria which may be effectively verified for the (local) uniqueness of the problem, and conditions sufficient for existence.

2. Reduction to an Auxiliary Problem. Let $\phi(x, y)$ be the adjoint function which is harmonic to ψ (friction potential). If we designate the (unknown) flow circulation by $\tilde{\nu} = \operatorname{sign} c_1 \int_\gamma Q\, ds \neq 0$, and if we investigate the function

$$\tau \equiv \tau(z) \equiv \exp\left\{\frac{2\pi i}{\tilde{\nu}} \chi(z)\right\}, \quad \chi = \varphi + i\psi, \quad \tilde{\nu} \neq 0, \tag{1}$$

we may readily show that it performs conformal and one-sheeted mapping of the unknown region G_z into the circle $G_\tau : r \leq |\tau| \leq 1$, where $r = \exp\{-(2\pi/\tilde{\nu})c_1\} < 1$ (the radius r is unknown). Therefore, the function $z = z(\tau)$, which is the inverse function of (1), is found. The last condition of the problem in terms of the variable $z(\tau)$ assumes the following form

$$\left|\frac{dz}{d\tau}\right| = \frac{\lambda}{Q(x,y)} \quad \text{for} \quad |\tau| = r, \quad \text{where} \quad \lambda = \frac{|\tilde{\nu}|}{2\pi} \exp\left\{\frac{2\pi |c_1|}{|\tilde{\nu}|}\right\}. \tag{2}$$

The derivative $dz/d\tau$ contains no zeros in $\overline{G_z}$. In addition, the function

$$F(\tau) = Ln \frac{dz(\tau)}{\lambda\, d\tau} = \ln\left|\frac{dz(\tau)}{\lambda\, d\tau}\right| + i \operatorname{Arg} \frac{dz(\tau)}{d\tau} \tag{3}$$

is single valued, regular, and analytic in G_τ. The function $z(\tau)$ can be readily expressed by means of $F(\tau)$. Performing the appropriate normalization in the selection of ϕ, we obtain the formula

$$z(\tau)=\lambda\int_{1}^{\tau}\exp\{F(t)\}\,dt+1. \tag{4}$$

Thus, the problem may be reduced to determining the function $F(\tau)$, /980 for which the function (4) is one-sheeted in G_τ.

3. Reduction to Integral Equations. Let us represent the function $F(\tau)$ in terms of the Will formula:

$$\begin{aligned} F(\tau)=&\frac{i}{\pi}\int_{0}^{2\pi}\mu(s)\zeta\left(\frac{1}{i}\ln\tau-s;\pi,-i\ln r\right)ds-\\ &-\frac{i}{\pi}\int_{0}^{2\pi}\mu_1(s)\zeta\left(\frac{1}{i}\ln\tau-s+i\ln r;\pi,-i\ln r\right)ds-\\ &-\left(\frac{1}{2}+\frac{\eta(r)}{\pi}\ln r\right)\frac{1}{\pi}\int_{0}^{2\pi}\mu_1(s)\,ds+iC,\quad \eta(r)=\frac{\pi}{12}\left[1-24\sum_{k=1}^{\infty}\frac{r^{2k}}{(1-r^{2k})^2}\right], \end{aligned} \tag{5}$$

where $\zeta(u;\ \omega,\ \omega')$ is the Weirstrass function with the half-periods $\omega=\pi$, $\omega'=-i\ \ln r$; C – the real constant [see, for example, (Ref. 1); we should point out that in this book, formula (5) is not given precisely. It does not include the last component on the right].

In formula (5) we have set $\mu(s)=\operatorname{Re}F(e^{is})$, $\mu_1(s)=\operatorname{Re}F(re^{is})$; the term iC is unimportant. Comparing formulas (3) and (2), we can see that $\mu_1(s)=-\ln Q(x,y)$, where $x(s)+iy(s)=z^+(re^{is})$. In order to make the function (5) single-valued, it is necessary and sufficient that the following condition be fulfilled

$$A_0\equiv\int_{0}^{2\pi}[\mu(s)-\mu_1(s)]\,ds=\int_{0}^{2\pi}[\mu(s)+\ln Q]\,ds=0. \tag{6}$$

For purposes of brevity, let us introduce the designation

$(\zeta(u)\ \equiv\ \zeta(u;\ \omega,\ \omega'))$

$$S_0\mu(\sigma) \equiv \frac{i}{\pi}\int_0^{2\pi} \mu(s)\zeta(\sigma - s)\,ds, \quad S\mu = \mu + S_0\mu;$$

$$S_1\mu(\sigma) = \frac{i}{\pi}\int_0^{2\pi} \mu(s)\left[\zeta(\sigma - s + i\ln r) - i\left(\frac{1}{2} + \frac{\eta}{\pi}\ln r\right)\right]ds,$$

$$S_2\mu(s) = \frac{i}{\pi}\int_0^{2\pi} \mu(s)\left[\zeta(\sigma - s - i\ln r) + i\left(\frac{1}{r} + \frac{\eta}{\pi}\ln r\right)\right]ds;$$

The first integral designates the principal Cauchy value. We shall assume that the functions $\mu(s)$, $x(s)$, $y(s)$ and the parameter λ are unknown. Let us substitute (5) in (4), and let us compute the limiting values of $z^+(e^{i\sigma})$. By requiring that the points of the circle $|\tau| = 1$ map into the points of the circle $|z| = 1$, we obtain the first equation

$$A_1 \equiv \left| i\lambda \int_0^{\sigma} \exp\{i\sigma + S\mu(\sigma) + S_1 \ln Q(x, y)(\sigma)\}\,d\sigma + 1\right|^2 - 1 = 0 \tag{7}$$

in order to determine λ, $\mu(s)$, $x(s)$, $y(s)$. We obtain two other equations by computing the values of $z^+(re^{is})$:

$$z^+(re^{i\sigma}) \equiv x(\sigma) + iy(\sigma) = 1 + \lambda\int_1^r \exp\{F(t)\}\,dt + $$
$$+ i\lambda r\int_0^{\sigma} Q^{-1}(x(\sigma), y(\sigma)) \exp\{i\sigma + S_0 \ln Q(x, y)(\sigma) + S_2\mu(\sigma)\}\,d\sigma. \tag{8}$$

Thus, the problem under consideration is equivalent to the system /981
of equations (7) and (8) for the unknowns λ, $\mu(s)$, $x(s)$, $y(s)$, and we must find its solution which makes the functions (4) and (5) one-sheeted.

The results presented below pertain to the special case when $Q(x, y) \equiv q(\rho)$, where $\rho^2 = x^2 + y^2$. In this case, λ, $\mu(s)$ and $\rho(s)$ are unknown. In order to determine them, we obtain equation (7) together with the equation

$$A_2 \equiv \rho^2(\sigma) - \left| 1 + \lambda L + i\lambda r\int_0^{\sigma} \frac{1}{q[\rho(\sigma)]}\exp\{i\sigma + S_2\mu + S_0 \ln q(\rho)(\sigma)\}\,d\sigma\right|^2 = 0 \tag{8'}$$

(in order to obtain this equation, we must take the modulus of both parts of (8) and square them; L is the function corresponding to the first integral in (8)).

4. Spaces and Operators. Let us designate the triplet set $\omega=(\lambda, \mu, \rho)$ by E, and let us assume that $\mu, \rho \in \mathrm{Lip}\,\beta, \beta > 0$, and let us assume that $\mu(s)$ is 2π-periodic. Introducing the norm $\| \omega \|_E = |\lambda| + \| \mu \|_{\mathrm{Lip}\,\beta} + \| \rho \|_{\mathrm{Lip}\,\beta}$ for E, we obtain the total, normalized, linear Banach space. The radius r and the parameter λ are related by the equation $\lambda r \ln r = - |c_1|$; thus, in the vicinity of any $r \neq e^{-1}$, the single-valued branch $r = r(\lambda)$ is determined. Continuing $r(\lambda)$ and $q(\rho)$ for all real λ and ρ, and maintaining sufficient smoothness, we can extend the operators A_0, A_1, A_2 to all E. We designate the triplet set $\omega_1 = (\mu_0, \mu_1, \mu_2)$ by E_1, where μ_0 is a number, μ_1, μ_2 are functions of s, and $\mu_2 \in \mathrm{Lip}\,\beta$, $\mu_1 \in \mathrm{Lip}\,\beta$, $\mu_1' \in \mathrm{Lip}\,\beta$, $\mu_1(0) = 0$. The set E_1, which is normalized by means of the formula $\| \omega_1 \|_{E_1} = |\mu_0| + \| \mu_1 \|_{\mathrm{Lip}\,\beta} + \| \mu_1' \|_{\mathrm{Lip}\,\beta} + \| \mu_2 \|_{\mathrm{Lip}\,\beta}$, becomes a Banach space. It is found that the three operators A_0, A_1, A_2 determine the continuous and continuously differentiable mapping $\omega_1 = \phi(\omega)E$ into E_1.

When attempting to determine the symmetrical region G_z: $|z| = \rho_0$, we obtain $\psi = c_1 \ln \rho / \ln \rho_0$ and ρ_0 is determined from equation

$$\rho_0 q(\rho_0) \ln \rho_0 = -|c_1|. \tag{9}$$

If ρ_0 is the root of this equation, then the triplet

$$\omega_0 = (\lambda_0, \mu_0, \rho_0) = (q(\rho_0); \ln q(\rho_0); \rho_0),$$

as may be seen, satisfies equation $\phi(\omega_0) = 0$. The problem consists of finding the solution of equation $\phi(\omega) = 0$ which is close to ω_0.

5. Linearized System. Theorem of Uniqueness. Let us set $X = (\Delta\lambda, h, l)$, $X_1 = (\mu_0, \mu_1, \mu_2)$, and let us investigate the inhomogeneous

equation

$$\varphi'(\omega_0;\ X) = X_1, \quad X \in E, \quad X_1 \in E_1, \tag{10}$$

where $\phi'(\omega_0;\ X)$ is the Freshe derivative of mapping $\omega_1 = \phi(\omega)$ at the point ω_0. Equation (10) is equivalent to a certain linear boundary problem in a circle for an analytical function. By expansion in Laurent series, this problem can be solved completely.

Theorem 1. Let $q(\rho)$ have a second derivative which is continuous in the sense of Hölder, and which is positive in the vicinity of the root ρ_0 of equation (9). In addition, let us set $q'(\rho_0) \neq 0$, $\rho_0 \neq e^{-1}$, and let us assume that the following condition is fulfilled

$$\begin{gathered}\frac{q(\rho_0)}{q'(\rho_0)}\rho_0^{n} + \frac{n-1}{n+1}\frac{q(\rho_0)}{q'(\rho_0)}\rho_0^{-n} - \frac{\rho_0^{1-n}}{1+n} + \frac{\rho_0^{1+n}}{1+n} \neq 0, \quad n = 1, 2, \ldots, \\ \frac{q(\rho_0)}{q'(\rho_0)} + \rho_0 - \frac{\rho_0}{1+\ln\rho_0} \neq 0.\end{gathered} \tag{11}$$

In order that equation (10) may be solved, it is necessary and sufficient that the right part X_1 satisfy the following condition

$$\eta(\rho_0)\mu_0 - H(\mu_1, \mu_2) = 0, \tag{12}$$

where $H(\mu_1, \mu_2)$ is a certain linear function (which can be clearly defined). Under conditions (11), the homogeneous equation $(X_1 = 0)$ has only the trivial solution. /982

Cases are also examined when $\rho_0 = e^{-1}$ or $q'(\rho_0) = 0$. If even one of conditions (11) is disturbed, non-trivial solutions, which may be clearly defined, appear for the homogeneous equation.

Theorem 2. (regarding local uniqueness). Under the conditions of theorem 1, the initial value problem in a certain (generally-speaking, small) vicinity of the function $\rho(s) = \rho_0$ (in the sense of metrics Lip β, $\beta > 0$) does not have non-trivial (i.e., different from circular) solutions.

The proof of this is based on the fact that under the conditions of theorem 1, the Freshe derivative $\phi'(\omega_0; X)$ contains the left inverse operator.

6. Existence Theorem. The problem we are considering encompasses the case of periodic waves in a heavy liquid. The corresponding function $q(\rho)$ has the form $q(\rho) = (2\pi\rho)^{-1} l \sqrt{C + \pi^{-1} g l \ln \rho}$, where l is the wave period, and C is the Bernoulli constant. Therefore, we shall assume that – in addition to the variable ρ – q contains a certain amount of parameters

$$\nu = (\nu_1, \nu_2, \ldots, \nu_n).$$

The mapping $\omega_1 = \phi(\omega; \nu)$ will also depend on ν. We shall use ρ_0 to designate the root of equation (9) for certain fixed values of $\nu = \nu^0$. Under the conditions of theorem 1, it is found that in the case of $\eta(\rho_0) \neq 0$ there is an n-parametric solution: $\lambda = \lambda(\nu)$, $\mu = \mu(\nu, s)$, $\rho = \rho(\nu, s)$ of the system of equations $A_1 = 0$, $A_2 = 0$, which is determined in the vicinity of (ρ_0, ν^0) and is such that $\lambda(\nu^0) = \lambda_0$, $\mu(\nu^0, s) = \mu_0$, $\rho(\nu^0, s) = \rho_0$.

In order that the function (4) be one-sheeted, it is necessary that

$$f_1(\nu_1, \nu_2, \ldots, \nu_n) = \operatorname{Im} \int_{|\tau|=1} \exp\{F(t, \nu)\}\, dt = 0, \tag{13}$$

where $F(t, \nu)$ is a function which is given by a formula such as (5) for a given solution. We should point out that $f_1(\nu_1^0, \nu_2^0, \ldots, \nu_n^0) = 0$, because in this case the function (1) coincides with z. Direct computations show that

$$\left.\frac{\partial f_1}{\partial \nu_1}\right|_{\nu=\nu^0} = \frac{[4\eta(\rho_0) q'_{\nu_1}(\rho_0, \nu^0)]}{\rho_0 q^2(\rho_0, \nu^0)} \frac{1-\rho_0}{\rho_0 q(\rho_0, \nu^0) / q'_\rho(\rho_0 \nu^0) + 1/2(\rho_0^2 - 1)} \times$$
$$\times \left[\frac{1}{1-\rho_0^2} + \frac{2\eta(\rho_0)}{\pi}(\rho_0 - 1)\right]. \tag{14}$$

It thus follows that if

$$q_\rho'(\rho_0, \nu^0) \neq 0, \quad q_{\nu_1}'(\rho_0, \nu^0) \neq 0, \quad \eta(\rho_0) \neq 0, \tag{15}$$

the partial derivative of the function (13) with respect to ν_1 at the initial point $\nu = \nu^0$ is different from zero. Consequently, there is a function $\nu_1 = \nu_1(\nu_2, \ldots, \nu_n)$, for which the function (13) vanishes. It can be also shown that the function $f_2(\nu) = A_0(\mu(s, \nu), \rho(s, \nu))$ vanishes for this function, and that the function (4) is one-sheeted.

Theorem 3. (existence theorem). Let the function q depend on ρ and on n (important) parameters $\nu_1, \nu_2, \ldots, \nu_n$, and let the conditions of theorem 1 be fulfilled at the point (ρ_0, ν^0). If q is continuously differentiable with respect to the parameters ν and if the condition (15) is fulfilled, then in a certain vicinity of $\rho = \rho_0$, the initial value problem has a $(n - 1)$-parametric set of solutions which are different from the trivial solution $\rho = \rho_0$.

We should point out that the variable c_1 from the third condition may also be assumed to be a parameter of the problem. In this case, there is a n-parametric set of solutions. As a consequence, we obtain the existence theorem of a biparametric set of periodic waves.

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